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Vibration of vertical rectangular plate in contact with water on one side

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SUMMARY

In this paper, the vibratory characteristics of a rectangular plate in contact with water on one side are studied. The elastic plate is considered to be a part of a vertical rectangular rigid wall in contact with water, the edges of which are elastically restrained and parallel to those of the rigid wall. The location and size of the plate on the rigid wall may vary arbitrarily. The water with a free surface is in a rectangular domain infinite in the length direction. The effects of free surface waves, compressibility of the water and the hydrostatic water pressure are neglected in the analysis. An analytical-Ritz method is developed to analyse the interaction of the plate–water system. First of all, by using the method of separation of variables and the method of Fourier series expansion, the exact expression of the motion of water is derived in the form of integral equations including the dynamic deformation of the plate. Then the Rayleigh–Ritz approach is used to derive the eigenfrequency equation of the system via the variational principle of energy. By selecting beam vibrating functions as the admissible functions of the plate, the added virtual mass incremental (AVMI) matrices for plate vibration are obtained. The convergency studies are carried out. The effects of some parameters such as the depth and width of water, the support stiffnesses, location and aspect ratio of the plate and the plate–water size and density ratios on the eigenfrequencies of the plate–water system are investigated. Several numerical examples are given. The validity of AVMI factor approach is also confirmed by comparing the AVMI factor solutions with the analytical-Ritz solutions. The results show that the approach presented here can also be used as excellent approximate solutions for rectangular plates in contact with water of infinite width and/or infinite depth. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: structural dynamics; fluid–plate interaction; hydroelastic dynamics, vibration; rectangular plate; analytical-Ritz method

INTRODUCTION

The problem of vibrating plates in contact with water has interested many investigators, both in the past [1–3] and at present [4–6]. This problem can be found in various kinds of engineering fields: such as containers, structures under or partially immersed in water, dams and floodgates,

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culvert cloughes, the hull and localized vibrations of ships and submarines, and so on. It is generally known that the eigenfrequencies of structures in contact with water decrease significantly compared to those in vacuum, especially for the fundamental eigenfrequency. Various methods have been used to resolve the fluid–structure interaction including analytical [7, 8], semi-analytical [9–11] and numerical methods [12, 13]. It is obvious that numerical methods such as finite element and boundary element can be generally applied to such problems, however, the modelling, code preparation and numerical computation require a long time. While the analytical method can give exact (or accurate) solutions to the problem addressed, it is however, invariably limited to very special and simple cases. Semi-analytical methods, which combine the advantages of wide applicability of numerical method and the high accuracy of analytical method, can solve the problem presented here at a small cost.

Up to now, most of the studies on the interaction of plates in contact with water is about circular plates. For example, non-dimensional added virtual mass incremental (NAVMI) factors for circular plates placed on a free fluid surface are computed by Kwak and Kim [14] and Kwak [15] using Hankel transformation for axisymmetric modes and all modes, respectively. Such a problem was further studied by Amabili and Kwak [5] using the Rayleigh–Ritz approach. Chiba [16] studied the linear axisymmetric vibration of a cylindrical tank with rigid wall and elastic bottom. Amabili [10] studied the vibrations of plates placed in a circular (or annular) aperture of an infinite rigid wall in contact with an unbounded fluid of finite depth on one side by using the Rayleigh–Ritz method. In comparison, studies on the interaction of rectangular plates in contact with liquid received less attention, partly because of the higher complexity of analysis for rectangular plates than for circular (or annular) plates. However, Bauer [7] studied the interaction of a simply supported elastic bottom of rectangular container having rigid wall and filled with liquid having a free surface by the exact analytical method. Marcus [12] studied the free vibration of vertical and horizontal cantilever plates submerged in an infinite water domain by the finite element method. Recently, Kwak [11] studied the NAVMI factors of a baffled rectangular plate in contact with water by the combined use of Rayleigh–Ritz method and Green's function method. Moreover, Soedel and Soedel [8] studied the free and forced vibration of a simply supported rectangular plate connected to a large reservoir of liquid at the four edges and supporting a liquid with freely sloshing surface.

In this paper, a more general case is considered for the interaction of a rectangular plate in contact with water: the plate is placed in a rectangular aperture of a vertical rectangular rigid wall in contact with water of finite width and depth, but infinite length. The edges of the plate are parallel to the edges of the wall and the location of the plate on the wall and the size of the plate may vary. Such a model can be used to analyse the hydroelastic vibration of floodgates, culvert cloughes and so on. An analytical-Ritz method is developed to analyse the vibratory characteristics of water–plate interaction. Convergency studies are carried out and some numerical results are given. It can be seen that the approach presented in this paper is also suitable for the interaction of rectangular plates in contact with water of infinite width (and/or infinite depth) which may be replaced by using equivalent finite, but, larger width (and/or larger depth) in the computation.

THE MODEL OF INVESTIGATION

Consider a rectangular plate with width a and height b, in contact with water on one side, as shown in Figure 1. The water is in a rectangular domain with finite width d and depth h, but with

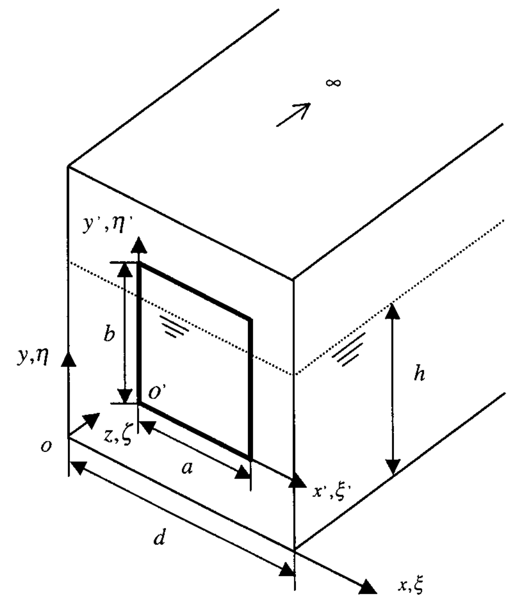

Figure 1. The sketch of the water–plate interaction system.

infinite length. The wall with the elastic plate stands at one end of the water domain in the length direction. It is assumed that the plate is thin and made of isotropic, homogeneous and linearly elastic material, so that the Kirchhoff theory of plate vibrations is applicable. The water is considered as an ideal, incompressible and inviscid liquid with small amplitude motion so that the potential of the velocity exists according to the linearized theory of small movement of fluid. Both the free surface and the rigid bottom of the water are orthogonal to the wall. The surface waves and hydrostatic pressure effects are neglected in the present study. As a consequence of these hypotheses, the kinetic energy can only be attributed to the water; therefore the sloshing modes of the water are not obtained by the present approach and only the bulging modes are investigated.

Two sets of Cartesian co-ordinate systems (o, x, y, z and o', x', y', z), whose corresponding axes are parallel to each other, are developed to describe the motion of the water and the vibration of the plate, respectively; o and o' are at the left bottom corner of the wall and the plate, respectively. The co-ordinates of the point o' in the o, x, y, z co-ordinate system are $x = x_0$, $y = y_0$, $z = 0$, which determines the location of the plate. The deformation of the plate in the z direction is defined by $w(x', y')$.

THE MOTION OF WATER

For an ideal, incompressible and inviscid liquid, the velocity potential $\phi(x, y, z, t)$ satisfies the following Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

where ϕ is defined by the velocity $v_x = -\partial\phi/\partial x$, $v_y = -\partial\phi/\partial y$ and $v_z = -\partial\phi/\partial z$, and ϕ should satisfy the boundary conditions

$$\frac{\partial\phi}{\partial x} = 0, \quad x = 0, d \tag{2}$$

$$\frac{\partial\phi}{\partial y} = 0, \quad y = 0 \tag{3}$$

$$\phi = 0, \quad y = h \tag{4}$$

$$\phi = 0, \quad z \to \infty \tag{5}$$

$$-\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \begin{cases} 0, & \text{the other part} \\ \dfrac{\partial w}{\partial t}, & x_0 \leqslant x \leqslant x_1, \quad y_0 \leqslant y \leqslant y_1 \end{cases} \tag{6}$$

where

$$x_1 = x_0 + a, \quad y_1 = \min\{h, y_0 + b\} \tag{7}$$

In the above equations, (2) and (3) imply the rigid wall conditions at the two opposite sides in the width direction and at the bottom of the water domain, respectively. Equation (4) means no surface wave on the free surface. Equation (5) implies no pressure disturbance is transmitted to infinity in the water domain and (6) implies that a part of the wall standing at the end of the water domain is elastic and the other part is rigid.

Applying the method of separation of variables, that is, by taking

$$\begin{aligned} w(x', y', t) &= W(x', y')\sin(\omega t) \\ \phi(x, y, z, t) &= X(x)Y(y)Z(z)\cos(\omega t) \end{aligned} \tag{8}$$

Equation (1) can be reduced to the following three uncoupled equations:

$$\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} \pm p_x^2 X &= 0 \\ \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} \pm p_y^2 Y &= 0 \\ \frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} \mp (p_x^2 + p_y^2) Z &= 0 \end{aligned} \tag{9}$$

where ω is the radian natural frequency of the plate–water system and p_x and p_y are arbitrary non-negative real numbers. The general solutions of the above equations can be obtained easily.

Considering the boundary equations (2)–(5), the solution of ϕ can be given as

$$\phi = \omega d\cos(\omega t)\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} A_{mj}\cos(m\pi\xi)\cos(\alpha_j\,\pi\eta)\,\mathrm{e}^{-\pi\sqrt{m^2+(\lambda\gamma\alpha_j/\beta)^2}\,\zeta} \tag{10}$$

in which, A_{mj} are the unknown constants and the following dimensionless co-ordinates and parameters are introduced:

$$\begin{aligned} &\xi = x/d, \quad \eta = y/h, \quad \zeta = z/d, \quad \alpha_j = j + 0.5, \quad \lambda = a/b \\ &\beta = a/d, \quad \gamma = b/h \end{aligned} \tag{11}$$

Substituting Equation (10) into (6) and applying the orthogonality of the trigonometric functions and the Fourier series expansion to the two sides of the equation in the rectangular domain: $0 \leqslant \xi \leqslant 1$ and $0 \leqslant \eta \leqslant 1$, the coefficients A_{mj} can be exactly derived in the form of integral equations as follows:

$$A_{mj} = \frac{\varepsilon_m}{q_{mj}} I_{mj} \tag{12}$$

in which

$$q_{mj} = \pi \sqrt{m^2 + (\lambda\gamma\alpha_j/\beta)^2}$$

$$\varepsilon_m = \begin{cases} 2, & m = 0 \\ 4, & m \neq 0 \end{cases} \tag{13}$$

$$I_{mj} = \int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} W(x', y') \cos(m\pi\xi) \cos(\alpha_j \pi\eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \tag{14}$$

where $\xi_1 = x_1/d = \xi_0 + \beta$ and $\eta_1 = y_1/h = \min\{1, \eta_0 + \gamma\}$, in which $\xi_0 = x_0/d$, and $\eta_0 = y_0/h$. Finally, the velocity potential of water is obtained as

$$\phi = \omega d \cos(\omega t) \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varepsilon_m}{q_{mj}} I_{mj} \cos(m\pi\xi) \cos(\alpha_j \pi\eta) \, \mathrm{e}^{-q_{mj}\zeta} \tag{15}$$

The integral equations I_{mj}, which represent the coupling effect of the plate and the water, will be dealt with later.

RAYLEIGH–RITZ SOLUTION

It is obvious that to arrive at an exact solution is impossible. Even for a rectangular plate in vacuum, the solutions cannot be exactly obtained except for the plate with two opposite edges simply supported. So approximate numerical methods must be used [17].

Kinetic energy of the water

Because the water is assumed as an ideal, incompressible and inviscid liquid and the free surface wave is also neglected, the kinetic energy can be only attributed to the water when the plate vibrates, which is expressed by T_w as follows:

$$T_w = \frac{1}{2} \rho_w \iiint_V (\nabla\phi)^2 \, \mathrm{d}v \tag{16}$$

where ρ_w is the density of water, V the water domain and $\nabla\phi$ the vector of velocity of water. By using Green's theorem for harmonic functions, the volume integration can be transformed into surface integration surrounding the water domain. Further considering the boundary conditions (2)–(5) of the water domain, one has

$$T_w = -\frac{1}{2}\rho_w \int_0^d \int_0^h \left(\phi \frac{\partial \phi}{\partial z}\right)\bigg|_{z=0} \mathrm{d}y\,\mathrm{d}x \tag{17}$$

Substituting Equations (6) and (15) into the above equation gives

$$(T_w)_{\max} = \frac{1}{2}\rho_w d^2 h\omega^2 \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{\varepsilon_m}{q_{mj}} I_{mj}^2 \tag{18}$$

Energy of the plate

It is well known that the plate can store both kinetic energy and potential energy, which are expressed by T_p and U_p, respectively, as follows:

$$(T_p)_{\max} = \frac{1}{2}\rho_p t_p \omega^2 \int_0^a \int_0^b W(x', y')^2\,\mathrm{d}y'\,\mathrm{d}x' \tag{19}$$

$$\begin{aligned}(U_p)_{\max} = \frac{1}{2} D \int_0^a \int_0^b \Bigg\{ & \left(\frac{\partial^2 W}{\partial x'^2}\right)^2 + 2\frac{\partial^2 W}{\partial x'^2}\frac{\partial^2 W}{\partial y'^2} + \left(\frac{\partial^2 W}{\partial y'^2}\right)^2 \\ & - 2(1-\nu)\left[\frac{\partial^2 W}{\partial x'^2}\frac{\partial^2 W}{\partial y'^2} - \left(\frac{\partial^2 W}{\partial x'\,\partial y'}\right)^2\right]\Bigg\}\mathrm{d}y'\,\mathrm{d}x' + \frac{1}{2}k_{tx0}\int_0^b W^2\bigg|_{x'=0}\mathrm{d}y' \\ & + \frac{1}{2}k_{tx1}\int_0^b W^2\bigg|_{x'=a}\mathrm{d}y' + \frac{1}{2}k_{ty0}\int_0^a W^2\bigg|_{y'=0}\mathrm{d}x' + \frac{1}{2}k_{ty1}\int_0^a W^2\bigg|_{y'=b}\mathrm{d}x' \\ & + \frac{1}{2}k_{rx0}\int_0^b \left(\frac{\partial W}{\partial x'}\right)^2\bigg|_{x'=0}\mathrm{d}y' + \frac{1}{2}k_{rx1}\int_0^b \left(\frac{\partial W}{\partial x'}\right)^2\bigg|_{x'=a}\mathrm{d}y' \\ & + \frac{1}{2}k_{ry0}\int_0^a \left(\frac{\partial W}{\partial y'}\right)^2\bigg|_{y'=0}\mathrm{d}x' + \frac{1}{2}k_{ry1}\int_0^a \left(\frac{\partial W}{\partial y'}\right)^2\bigg|_{y'=b}\mathrm{d}x'\end{aligned} \tag{20}$$

where ρ_p is the density of the plate material, ν the Poisson's ratio, t_p the plate thickness, D the flexural rigidity of the plate. k_{tx0} and k_{tx1} are the translational stiffnesses per unit length along the edges $x' = 0$ and a, respectively. k_{ty0} and k_{ty1} are those along the edges $y' = 0$ and b, respectively. k_{rx0} and k_{rx1} are the rotational stiffnesses per unit length along the edges $x' = 0$ and a, respectively. k_{ry0} and k_{ry1} are those along the edges $y' = 0$ and b, respectively.

Eigenfrequency equation

It is easily seen that the total kinetic energy T of the plate–water system is

$$(T)_{\max} = (T_p)_{\max} + (T_w)_{\max} \tag{21}$$

The total potential energy U of the system is only that of the plate, so the total energy E of the system is

$$E = (T)_{\max} - (U_{\mathrm{p}})_{\max} = (T_{\mathrm{p}})_{\max} + (T_{\mathrm{w}})_{\max} - (U_{\mathrm{p}})_{\max} \tag{22}$$

A series is assumed for the mode shape function $W(x', y')$ of the form

$$W(x', y') = \sum_{n=1}^{N} \sum_{l=1}^{L} C_{nl} f_n(x')\, g_l(y') \tag{23}$$

where C_{nl} are the unknown constants, N and L the truncated orders of the series, $f_n(x')$ and $g_l(y')$ the admissible functions, respectively, in the x' and y' directions.

Substituting Equation (23) into (22) and using the variational principle of energy, one has

$$\frac{\partial E}{\partial C_{nl}} = 0, \quad n = 1,2, \ldots, N, \quad l = 1,2, \ldots, L \tag{24}$$

which results in

$$\sum_{n=1}^{N} \sum_{l=1}^{L} [K_{nl\bar{n}\bar{l}} - \Omega^2 (M_{nl\bar{n}\bar{l}} + \tilde{M}_{nl\bar{n}\bar{l}})]\, C_{nl} = 0$$
$$\bar{n} = 1,2, \ldots, N, \quad \bar{l} = 1,2, \ldots, L \tag{25}$$

where

$$\begin{aligned} K_{nl\bar{n}\bar{l}} &= E_{n\bar{n}}^{(2,2)} F_{l\bar{l}}^{(0,0)} + \lambda^4 E_{n\bar{n}}^{(0,0)} F_{l\bar{l}}^{(2,2)} + \nu\lambda^2 (E_{n\bar{n}}^{(0,2)} F_{l\bar{l}}^{(2,0)} + E_{n\bar{n}}^{(2,0)} F_{l\bar{l}}^{(0,2)}) \\ &\quad + 2(1-\nu)\lambda^2 E_{n\bar{n}}^{(1,1)} F_{l\bar{l}}^{(1,1)} + \sum_{i=0}^{1} (K_{\mathrm{t}xi} P_{n\bar{n}}^{i} + K_{\mathrm{r}xi} R_{n\bar{n}}^{i})\, F_{l\bar{l}}^{(0,0)} \\ &\quad + \gamma^4 \sum_{i=0}^{1} (K_{\mathrm{t}yi} Q_{l\bar{l}}^{i} + K_{\mathrm{r}yi} S_{l\bar{l}}^{i})\, E_{n\bar{n}}^{(0,0)}, \quad \Omega^2 = \rho_{\mathrm{p}} t_{\mathrm{p}} \omega^2 a^4 / D \end{aligned}$$

$$M_{nl\bar{n}\bar{l}} = E_{n\bar{n}}^{(0,0)} F_{l\bar{l}}^{(0,0)}, \quad \tilde{M}_{nl\bar{n}\bar{l}} = \frac{\mu\lambda}{\sigma\beta^2\gamma} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varepsilon_m}{q_{mj}} \tilde{E}_{nm} \tilde{E}_{\bar{n}m} \tilde{F}_{lj} \tilde{F}_{\bar{l}j} \tag{26}$$

in which,

$$\begin{aligned} E_{n\bar{n}}^{(r,s)} &= \int_0^1 (\mathrm{d}^r f_n(\xi')/\mathrm{d}\xi'^r)(\mathrm{d}^s f_{\bar{n}}(\xi')/\mathrm{d}\xi'^s)\, \mathrm{d}\xi' \\ F_{l\bar{l}}^{(r,s)} &= \int_0^1 (\mathrm{d}^r g_l(\eta')/\mathrm{d}\eta'^r)(\mathrm{d}^s g_{\bar{l}}(\eta')/\mathrm{d}\eta'^s)\, \mathrm{d}\eta; \quad r, s = 0, 1, 2 \\ \tilde{E}_{nm} &= \int_{\xi_0}^{\xi_1} f_n(\xi') \cos(m\pi\xi)\, \mathrm{d}\xi, \quad \tilde{F}_{lj} = \int_{\eta_0}^{\eta_1} g_l(\eta') \cos(\alpha_j \pi\eta)\, \mathrm{d}\eta \\ P_{n\bar{n}}^{i} &= f_n(i)\, f_{\bar{n}}(i), \quad Q_{l\bar{l}}^{i} = g_l(i) g_{\bar{l}}(i) \\ R_{n\bar{n}}^{i} &= [\mathrm{d}f_n(\xi')/\mathrm{d}\xi' \;\; \mathrm{d}f_{\bar{n}}(\xi')/\mathrm{d}\xi']_{\xi'=\mathrm{i}} \\ S_{l\bar{l}}^{i} &= [\mathrm{d}g_l(\eta')/\mathrm{d}\eta' \; \mathrm{d}g_{\bar{l}}(\eta')/\mathrm{d}\eta']_{\eta'=i} \\ K_{\mathrm{t}xi} &= k_{\mathrm{t}xi}\, a^3/D, \quad K_{\mathrm{t}yi} = k_{\mathrm{t}yi}\, b^3/D \\ K_{\mathrm{r}xi} &= k_{\mathrm{r}xi}\, a/D, \quad K_{\mathrm{r}yi} = k_{\mathrm{r}yi}\, b/D, \quad i = 0, 1 \end{aligned} \tag{27}$$

In the above equations, the following dimensionless parameters and variables are introduced:

$$\mu = \rho_{\mathrm{w}}/\rho_p, \quad \sigma = t_{\mathrm{p}}/b, \quad \xi' = x'/a, \quad \eta' = y'/b \tag{28}$$

In the analysis of fluid–solid interaction, the matrix $\tilde{M}$ made up of $\tilde{M}_{nl\bar{n}\bar{l}}$ is called the added virtual mass incremental (AVMI) matrix [10], which means that the effect of water on the dynamic behaviour of the plate is equivalent to a generalized distributed mass attached to the plate. Resolving Equation (25) by the standard eigenvalue process, dimensionless eigenfrequencies $\Omega_i (i = 1, 2, \ldots, N \times L)$ and coefficients $C_{nl} (n = 1, 2, \ldots, N,\ l = 1, 2, \ldots, L)$ corresponding to each Ω_i can be obtained. Substituting the results into Equation (23) gives the mode shape corresponding to Ω_i. It should be mentioned that when calculating $\tilde{M}_{nl\bar{n}\bar{l}}$ numerically, the relations $\xi' = (\xi - \xi_0)/\beta$, $\eta' = (\eta - \eta_0)/\gamma$ was employed and the summing series about m and j should be truncated.

For the rectangular plate considered here, the beam vibrating functions are adopted as the admissible functions, which can be written, in general form, as follows:

$$f_n(\xi') = a_n \sin(k_n \xi') + b_n \cos(k_n \xi') + c_n \sinh(k_n \xi') + d_n \cosh(k_n \xi') \tag{29}$$

in which, the constants a_n, b_n, c_n, d_n and the eigenvalues k_n may be decided by the boundary conditions (corresponding to those of the plate) of the beam as follows:

$$\frac{\mathrm{d}^3 f_n(\xi')}{\mathrm{d}\xi'^3} = -(-1)^i K_{\mathrm{tx}i} f_n(\xi'), \quad \frac{\mathrm{d}^2 f_n(\xi')}{\mathrm{d}\xi'^2} = (-1)^i K_{\mathrm{rx}i} \frac{\mathrm{d} f_n(\xi')}{\mathrm{d}\xi'}, \quad \xi' = i, \quad i = 0, 1 \tag{30}$$

Similarly, $g_l(\eta')$ can also be given.

NUMERICAL EXAMPLES

In order to demonstrate the applicability and accuracy of the proposed technique, Equation (25) has been used to generate results for rectangular plates with a Poisson's ratio $v = 0.3$, thickness ratio $\sigma = 0.05$ and plate–water density ratio $\mu = 0.125$. First of all, convergency studies have been carried out for simply supported and fully clamped square plates ($\lambda = 1$) placed at the center ($\xi_0 = \eta_0 = (1 - \beta)/2$) of the square rigid wall ($\beta = \gamma$) for different plate-water size ratios, as tabulated in Tables I and II, respectively. From the two tables, it can be observed that the convergent speed is very rapid and the three or four beam vibrating functions give the first six eigenfrequencies of plate–water interaction with sufficiently high accuracy. It is also shown that the number of terms of the beam vibrating functions is basically unaffected by the plate–water size ratios, thus ensuring a small size matrix of eigenfrequency equation for all cases. Moreover, it can be seen that the truncated orders j and m of the double series in the eigenfrequency equation are concerned with the plate–water size ratios and will increase with a decrease of the plate–water size ratios. Furthermore, eigenfrequencies of plate–water interaction tend to take up constant values with the decrease of the plate–water size ratios, especially for the higher order of eigenfrequencies. This implies that the results for plates in contact with water of infinite width and/or infinite depth can be approximated by those in which the water is finite but with larger width and/or larger depth.

Table I. The convergency of eigenfrequencies of a simply supported square plate for different plate–water size ratios.

$N = L$	$j = m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$\beta = \gamma = 1$							
1	30	13.579					
2	30	13.564	39.955	40.418	67.709		
3	30	13.558	39.953	40.385	67.695	84.188	86.508
4	10	13.558	39.945	40.382	67.685	84.176	86.497
4	20	13.558	39.944	40.381	67.683	84.170	86.493
4	30	13.558	39.944	40.381	67.683	84.169	86.493
$\beta = \gamma = 1/2$							
1	30	13.756					
2	30	13.755	40.797	40.826	68.422		
3	30	13.751	40.795	40.823	68.422	84.790	86.788
4	10	13.752	40.803	40.832	68.450	84.842	86.850
4	20	13.751	40.793	40.821	68.419	84.792	86.791
4	30	13.751	40.792	40.820	68.417	84.789	86.788
$\beta = \gamma = 1/4$							
1	40	13.819					
2	40	13.819	40.889	40.891	68.443		
3	40	13.815	40.887	40.890	68.443	84.880	86.803
4	10	13.830	41.463	41.500	70.200	90.807	93.581
4	20	13.815	40.895	40.898	68.472	84.933	86.864
4	30	13.815	40.886	40.888	68.442	84.888	86.813
4	40	13.815	40.885	40.887	68.439	84.880	86.803
$\beta = \gamma = 1/8$							
1	50	13.848					
2	50	13.848	40.904	40.904	68.456		
3	50	13.844	40.903	40.903	68.456	84.936	86.827
4	20	13.856	41.494	41.505	70.234	90.614	93.049
4	30	13.848	40.926	40.928	68.519	85.619	87.608
4	40	13.844	40.907	40.907	68.473	84.968	86.865
4	50	13.844	40.900	40.900	68.451	84.933	86.824
Plate in vacuum							
exact		19.739	49.348	49.348	78.957	98.696	98.696

It should be pointed out that in Equation (25), if the admissible functions used are the exact modal solutions of the plate in vacuum and the AVMI matrix $\tilde{M}$ is diagonal, namely, the modes of the plate–water system will be the same as those of the plate in vacuum. In this case, the analysis will be greatly simplified and the dimensionless eigenfrequencies of the plate–water interaction can be written as $\Omega_i = \bar{\Omega}_i/\sqrt{1 + \tilde{M}_{ii}/M_{ii}}$, where $\bar{\Omega}_i$ are the dimensionless eigenfrequencies of the plate in vacuum, M_{ii} and $\tilde{M}_{ii}$ (called as AVMI factors) are the ith diagonal elements of matrices M and $\tilde{M}$, respectively. The AVMI matrices for a simply supported square plate with different plate–water size ratios are given, by using two beam vibrating functions in

Table II. The convergency of eigenfrequencies of a fully clamped square plate for different plate–water size ratios.

$N=L$	$j=m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$\beta=\gamma=1$							
1	30	25.981					
2	30	25.958	61.363	61.769	95.091		
3	30	25.775	61.070	61.424	95.086	112.87	116.73
4	10	25.773	60.981	61.352	94.460	112.83	116.72
4	20	25.773	60.981	61.352	94.460	112.83	116.72
4	30	25.773	60.981	61.352	94.460	112.83	116.72
$\beta=\gamma=1/2$							
1	30	26.216					
2	30	26.215	62.156	62.182	95.608		
3	30	26.042	61.874	61.898	95.608	113.44	116.99
4	10	26.043	61.820	61.846	95.077	113.57	117.13
4	20	26.042	61.807	61.832	95.031	113.44	116.99
4	30	26.042	61.807	61.832	95.030	113.44	116.99
$\beta=\gamma=1/4$							
1	40	26.307					
2	40	26.307	62.244	62.246	95.623		
3	40	26.136	61.964	61.966	95.623	113.58	117.00
4	10	26.228	63.393	63.484	98.785	122.37	126.82
4	20	26.137	61.911	61.914	95.091	113.66	117.08
4	30	26.136	61.899	61.901	95.049	113.58	117.00
4	40	26.136	61.898	61.901	95.047	113.58	117.00
$\beta=\gamma=1/8$							
1	50	26.350					
2	50	26.350	62.256	62.256	95.628		
3	50	26.180	61.977	61.977	95.628	113.66	117.03
4	20	26.255	63.430	63.466	98.821	122.13	126.31
4	30	26.184	61.975	61.977	95.209	115.49	119.08
4	40	26.181	61.922	61.922	95.091	113.71	117.08
4	50	26.180	61.911	61.912	95.052	113.66	117.02
Plate in vacuum							
4		36.007	73.460	73.460	108.37	131.77	132.33

each direction, as follows:

$$\tilde{M}=\begin{bmatrix} 2.783\times10^{-1} & -1.493\times10^{-17} & 5.867\times10^{-2} & -2.951\times10^{-18} \\ -1.493\times10^{-17} & 1.311\times10^{-1} & -2.951\times10^{-18} & 1.185\times10^{-2} \\ 5.867\times10^{-2} & -2.951\times10^{-18} & 1.300\times10^{-1} & -7.270\times10^{-18} \\ -2.951\times10^{-18} & 1.185\times10^{-2} & -7.270\times10^{-18} & 9.052\times10^{-2} \end{bmatrix} \tag{31}$$

for $\beta = \gamma = 1$ and

$$\tilde{M} = \begin{bmatrix} 2.580\times10^{-1} & 1.897\times10^{-17} & 8.151\times10^{-4} & -3.378\times10^{-20} \\ 1.897\times10^{-17} & 1.139\times10^{-1} & -3.378\times10^{-20} & 1.324\times10^{-6} \\ 8.151\times10^{-4} & -3.378\times10^{-20} & 1.139\times10^{-1} & 1.157\times10^{-17} \\ -3.378\times10^{-20} & 1.324\times10^{-6} & 1.157\times10^{-17} & 8.259\times10^{-2} \end{bmatrix} \tag{32}$$

for $\beta = \gamma = 1/8$. It can be seen that AVMI matrix $\tilde{M}$ for the simply supported square plate–water system is diagonal dominant (namely, $\tilde{M}_{ii} \gg \tilde{M}_{ij}$, $i \neq j$) especially for $\beta = \gamma = 1/8$ (larger water domain). This implies that if the diagonal elements of $\tilde{M}$ are taken as the AVMI factors [1, 14], the errors between the AVMI factor solutions $\tilde{\Omega}_i \approx \bar{\Omega}_i/\sqrt{1 + 4\tilde{M}_{ii}}$ and the more accurate analytical-Ritz solutions Ω_i will be rather small and tend to decrease with an increase of the width and the depth of the water. The same conclusion can also be obtained for the fully clamped square plate–water system. The AVMI matrices for the fully clamped square plate with different plate–water size ratios are given, for the two beam vibrating functions in each direction as follows:

$$\tilde{M} = \begin{bmatrix} 9.316\times10^{-1} & -4.173\times10^{-8} & 1.150\times10^{-1} & -2.204\times10^{-9} \\ -4.173\times10^{-8} & 4.437\times10^{-1} & -2.204\times10^{-9} & 2.415\times10^{-2} \\ 1.150\times10^{-1} & -2.204\times10^{-9} & 4.394\times10^{-1} & -2.925\times10^{-8} \\ -2.204\times10^{-9} & 2.451\times10^{-2} & -2.925\times10^{-8} & 3.110\times10^{-1} \end{bmatrix} \tag{33}$$

for $\beta = \gamma = 1$ and

$$\tilde{M} = \begin{bmatrix} 8.779\times10^{-1} & -3.943\times10^{-8} & 2.116\times10^{-3} & -5.897\times10^{-13} \\ -3.943\times10^{-8} & 4.028\times10^{-1} & -5.897\times10^{-13} & 6.183\times10^{-6} \\ 2.116\times10^{-3} & -5.897\times10^{-13} & 4.028\times10^{-1} & -2.819\times10^{-8} \\ -5.897\times10^{-13} & 6.183\times10^{-6} & -2.819\times10^{-8} & 2.957\times10^{-1} \end{bmatrix} \tag{34}$$

for $\beta = \gamma = 1/8$. Comparing Equations (31) and (33), (32) and (34), respectively, one can find that the AVMI matrices for the simply supported plate are more diagonal dominant than those for the fully clamped plate. However, It should be mentioned that because the exact modes for the fully clamped plate in vacuum cannot be given, the approximate modal solutions should be used, which results in the non-diagonal stiffness matrix K. In this case, even if the off-diagonal elements in $\tilde{M}$ are neglected, a standard eigenvalue computation is still necessary to obtain $\tilde{\Omega}$. The percentage errors $e_i(\%) = (1 - \tilde{\Omega}_i/\Omega_i) \times 100$ $(i = 1,2,3,4)$ between the AVMI factor solutions and the analytical-Ritz solutions for the first four eigenfrequencies are shown in Table III for the simply supported and fully clamped square plates with different plate–water size ratios. From the table, one can see that the errors are very small and the maximum error is less than 1 per cent. This shows that the AVMI factor approach is applicable to the rectangular plate–water system considered here as well as to the circular and annular plate–water system [10, 14].

Table III. The percentage errors $e_i(\%)$ ($i = 1,2,3,4$) of the AVMI factor solutions with respect to the analytical-Ritz solutions for the first four eigenfrequencies of the plate–water system.

$\beta = \gamma$	e_1 (%)	e_2 (%)	e_3 (%)	e_4 (%)
SSSS				
1	− 0.155	0.058	0.882	0.044
1/2	− 0.036	− 0.012	0.039	− 0.006
1/4	− 0.029	− 0.010	− 0.007	− 0.006
1/8	− 0.029	− 0.010	− 0.010	− 0.007
CCCC				
1	− 0.525	− 0.271	− 0.189	− 0.212
1/2	− 0.392	− 0.197	− 0.170	− 0.178
1/4	− 0.375	− 0.191	− 0.187	− 0.176
1/8	− 0.371	− 0.189	− 0.187	− 0.175

The first six dimensionless eigenfrequencies for a square plate ($\lambda = 1$) and a rectangular plate ($\lambda = 0.5$) with five kinds of classical boundary conditions: SSSS, CCCC, SCSF, CCCF, SCSC, partially in contact with water on one side are tabulated in Tables IV and V, respectively. The width of the water is the same as that of the plate. The results show that the eigenfrequencies of the plate–water system are always lower than those of the plate in vacuum and the eigenfrequencies of the plate–water interaction decrease with the increase of the depth of water.

When a square plate on a square rigid wall is entirely immersed in water on one side, the effect of the location of the plate on the eigenfrequencies of the plate–water system is studied and the first six dimensionless eigenfrequencies are tabulated in Table VI. The width and height of the plate are a quarter of those of water, respectively. It can be seen that the closer the plate is to the left bottom corner (or the right bottom corner) of the wall, the lower are the eigenfrequencies; the closer the plate is to the free surface of the water and the closer the plate is to the middle of the water in the width direction, the higher are the eigenfrequencies.

The effects of the width of water on the first two eigenfrequencies of the square plate–water interaction are given, respectively, in Figures 2 and 3 where the depth of the water is the same as the height of the plate. Four kinds of classical boundary conditions for the plate are considered. It can be seen that an increase in width of the water will result in lower eigenfrequencies, which however will rapidly approach constant values. This once again shows that the solutions for the plates in contact with water of somewhat larger finite width can be used as excellent approximate solutions for the plates in contact with water of infinite width.

The effects of the depth of water on the first two eigenfrequencies of the simply supported and fully clamped square plate are studied. The ratios $r_i = \Omega_i/\bar{\Omega}_i$ ($i = 1,2$) of the first two eigenfrequencies of the plate–water system with respect to those of the plate in vacuum are given in Figure 4. The plate is entirely under water and the width of the water is the same as that of the plate. It can be seen that the deeper water will result in lower eigenfrequencies which however, will also approach constant values, especially for higher modes. Comparing Figures 4 with 2, one can

Table IV. The first six dimensionless eigenfrequencies of a square plate partially in contact with water on one side.

γ	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SSSS						
0.0	19.739	49.348	49.348	78.957	98.696	98.696
0.2	19.666	48.734	49.177	78.050	96.690	98.382
0.4	18.451	44.763	46.650	73.378	92.322	94.311
0.6	16.854	43.721	43.911	72.703	89.436	90.867
0.8	14.875	41.193	42.141	70.287	86.180	89.249
1.0	13.558	39.944	40.381	67.683	84.169	86.493
CCCC						
0.0	36.007	73.460	73.460	108.37	131.77	132.33
0.2	35.968	73.089	73.374	107.83	130.52	131.93
0.4	34.216	67.602	70.256	101.81	124.00	126.95
0.6	31.273	66.004	66.167	100.83	120.51	122.15
0.8	27.795	62.347	63.319	97.096	115.84	119.94
1.0	25.773	60.981	61.352	94.460	112.83	116.72
SCSF						
0.0	12.756	33.091	41.992	63.192	72.422	91.192
0.2	12.746	32.779	41.943	62.482	70.312	91.069
0.4	12.677	31.389	41.641	59.994	66.678	90.335
0.6	12.401	29.069	40.671	57.241	65.153	88.219
0.8	11.504	27.327	38.447	56.263	63.045	84.454
1.0	10.047	26.324	35.662	54.655	62.087	80.312
CCCF						
0.0	24.035	40.085	63.545	76.869	80.819	117.06
0.2	24.033	40.036	63.536	76.473	80.691	116.43
0.4	23.873	38.044	62.967	71.072	76.864	110.83
0.6	23.294	35.471	61.311	69.633	74.022	109.81
0.8	21.505	33.836	58.049	67.362	73.243	106.34
1.0	19.000	32.538	54.571	66.342	70.793	104.88
SCSC						
0.0	28.957	54.794	69.332	94.649	102.35	129.11
0.2	28.926	54.730	68.981	94.166	102.22	127.65
0.4	27.491	52.359	63.573	88.426	98.271	121.15
0.6	25.016	49.058	61.907	87.389	93.654	118.60
0.8	22.016	45.805	59.411	84.202	89.449	116.83
1.0	20.262	44.612	57.531	81.741	87.900	112.69

find that the effect of the depth of water on the fundamental eigenfrequency of the plate–water system is larger than that for the width of water.

Finally, the effects of the support stiffnesses of the plate on eigenfrequencies of the plate–water system are investigated. The support stiffnesses of the plate are taken as $K_{tx0} = K_{tx1} = K_{ty0} = K_{ty1} = 10\ K_r$ and $K_{rx0} = K_{rx1} = K_{ry0} = K_{ry1} = K_r$. The width and depth of the water are

Table V. The first six dimensionless eigenfrequencies of a rectangular plate ($\lambda = 0.5$) partially in contact with water on one side.

γ	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SSSS						
0.0	12.333	19.739	32.076	41.946	49.348	49.348
0.2	12.292	19.499	31.443	41.812	48.253	48.847
0.4	11.494	18.032	30.186	39.963	46.951	47.099
0.6	10.481	17.695	29.483	38.465	46.592	46.881
0.8	9.3066	17.018	29.131	37.261	45.324	45.579
1.0	8.5151	16.306	28.130	36.802	44.142	44.643
CCCC						
0.0	24.583	31.856	44.817	63.431	64.002	71.188
0.2	24.550	31.687	44.313	62.447	63.904	70.821
0.4	23.149	29.156	42.465	60.597	61.256	68.572
0.6	21.152	29.196	41.576	59.110	60.106	68.156
0.8	19.056	27.880	40.991	57.566	58.957	65.777
1.0	17.723	26.981	39.649	57.074	57.694	64.554
SCSF						
0.0	10.486	15.778	25.838	40.024	40.640	45.238
0.2	10.484	15.750	25.695	40.013	40.173	45.135
0.4	10.378	14.813	24.018	38.464	39.528	43.212
0.6	10.026	13.850	23.604	37.731	38.447	42.560
0.8	9.1168	13.362	22.819	37.014	37.190	42.347
1.0	8.0512	12.813	22.479	35.731	36.438	40.888
CCCF						
0.0	22.710	26.068	33.839	46.814	62.077	65.355
0.2	22.705	26.019	33.652	46.291	62.060	64.492
0.4	22.337	24.510	31.800	44.551	60.879	62.932
0.6	20.927	23.800	31.329	43.725	58.967	62.476
0.8	18.851	23.489	30.289	43.071	57.277	61.344
1.0	17.208	22.102	29.812	42.244	56.003	60.269
SCSC						
0.0	13.689	23.662	38.708	42.606	51.781	58.678
0.2	13.671	23.536	38.261	42.542	51.509	57.730
0.4	12.957	21.783	36.462	40.875	49.436	55.907
0.6	11.803	21.292	35.700	39.188	49.191	55.451
0.8	10.443	20.430	35.202	37.841	47.484	54.354
1.0	9.6102	19.776	33.985	37.431	46.457	53.164

the same as the sizes of the plate. The first six eigenfrequencies for a square plate are given in Figure 5. It is shown that the support stiffnesses of the plate have an important effect on eigenfrequencies of the plate–water interaction. The eigenfrequencies increase with the increase of support stiffnesses of the plate and approach those of the clamped square plate when K_r approaches infinity.

Table VI. The first six eigenfrequencies of a square plate entirely immersed in water and at different locations on the vertical rigid wall.

ξ_0	η_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SSSS							
0	0	12.191	39.984	40.147	67.580	82.835	86.221
	1/4	13.024	40.394	40.568	68.043	83.816	86.516
	1/2	13.428	40.400	40.611	68.045	84.236	86.531
	3/4	14.557	40.717	41.287	68.535	85.590	87.146
1/8	0	12.710	40.379	40.509	68.032	83.498	86.500
	1/4	13.462	40.830	40.848	68.426	84.469	86.792
	1/2	13.866	40.845	40.879	68.430	84.929	86.794
	3/4	14.853	41.172	41.457	68.840	86.278	87.285
1/4	0	12.885	40.396	40.560	68.043	83.679	86.512
	1/4	13.597	40.866	40.873	68.435	84.627	86.798
	1/2	13.977	40.886	40.897	68.438	85.067	86.800
	3/4	14.893	41.198	41.475	68.846	86.334	87.289
3/8	0	12.932	40.398	40.570	68.044	83.727	86.514
	1/4	13.633	40.874	40.875	68.436	84.669	86.799
	1/2	14.003	40.894	40.898	68.439	85.099	86.801
	3/4	14.900	41.201	41.479	68.847	86.343	87.290
CCCC							
0	0	23.609	61.014	61.191	94.389	110.41	116.55
	1/4	24.928	61.414	61.601	94.746	111.97	116.78
	1/2	25.561	61.420	61.645	94.747	112.69	116.79
	3/4	27.239	61.716	62.280	95.115	114.94	117.22
1/8	0	24.431	61.399	61.542	94.737	111.42	116.77
	1/4	25.598	61.844	61.863	95.037	112.91	116.99
	1/2	26.215	61.859	61.893	95.040	113.67	116.99
	3/4	27.657	62.165	62.437	95.348	115.76	117.34
1/4	0	24.709	61.416	61.594	94.746	111.73	116.78
	1/4	25.804	61.880	61.887	95.044	113.17	117.00
	1/2	26.383	61.900	61.911	95.047	113.89	117.00
	3/4	27.714	62.191	62.455	95.353	115.85	117.34
3/8	0	24.783	61.418	61.603	94.746	111.81	116.78
	1/4	25.859	61.889	61.889	95.045	113.23	117.00
	1/2	26.422	61.908	61.912	95.047	113.94	117.00
	3/4	27.724	62.194	62.459	95.353	115.86	117.34

CONCLUSIONS

The analytical-Ritz approach is developed to analyse the vibration of a plate-water system. The plate with elastic edge constraints is placed in a rectangular aperture of a vertical rectangular

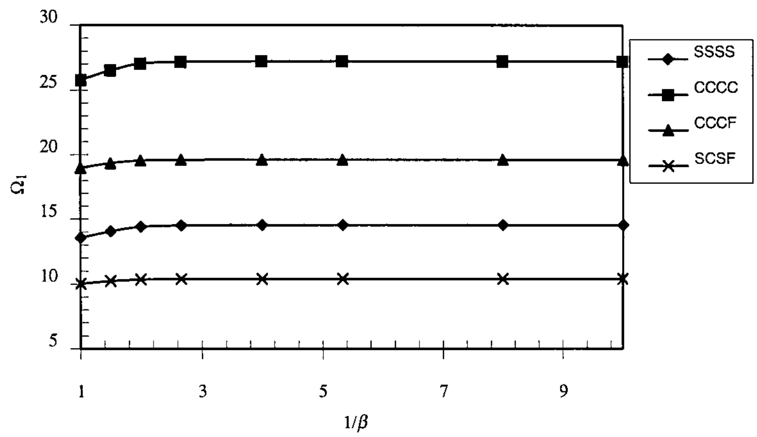

Figure 2. The fundamental eigenfrequency of a square plate with different boundary conditions and in contact with water of different width on one side.

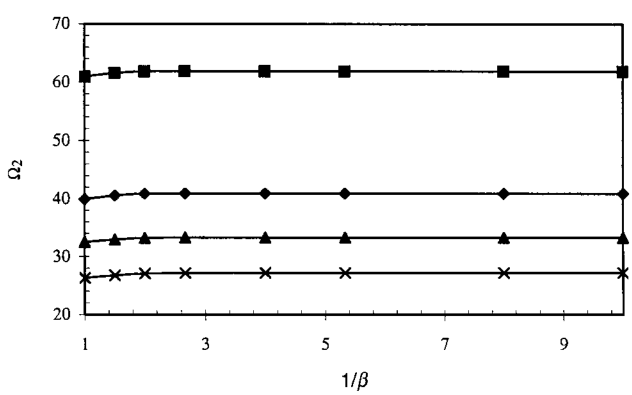

Figure 3. The second order of eigenfrequency of a square plate with different boundary conditions and in contact with water of different width on one side. Key as Figure 2.

rigid wall and is in contact with water on one side. The water domain is limited by a free surface and a rigid bottom in the depth direction and two opposite rigid walls in the width direction, but remains unlimited in the length direction. A rigid wall carrying the elastic plate stands at one end of the water domain in the length direction. The free vibration of the plate–water system is investigated. Convergency studies demonstrate the high accuracy and small computational cost of the method presented in the paper. The effects of some size ratios of the plate and water on the eigenfrequencies of the plate–water system are discussed in detail and some interesting conclusions are obtained. It is shown that the approach of AVMI factor is also applicable to the plate–water system considered here. The solutions for the rectangular plate in contact with water

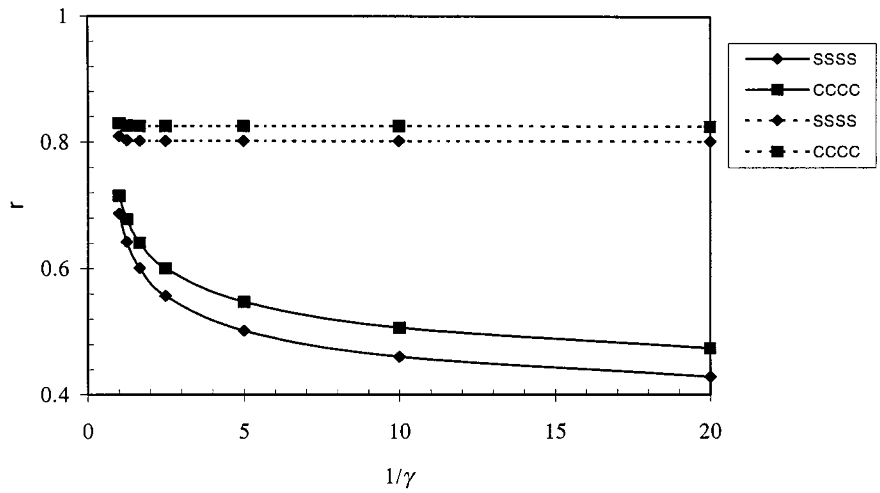

Figure 4. The ratios of the first two eigenfrequencies of simply supported and fully clamped square plates entirely immerged in water with respect to those of the plates in vacuum as a function of water depth. (———) r_1; (.............) r_2.

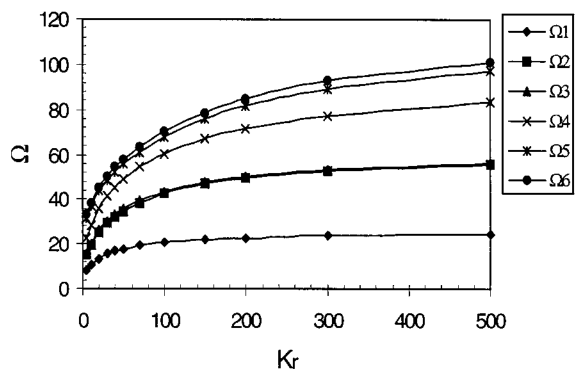

Figure 5. The first six plate–water eigenfrequencies for a square plate elastically supported on all edges and in contact with water having the same size as those of the plate.

of infinite width (and/or depth) can be approximately expressed by those for the plate in contact with water of finite, but, larger width (and/or depth) with sufficiently high accuracy.

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